Assignment # 5	Due on December 4th in class

1. On a finite dimensional (real) vector space X, assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on it. Prove that there exist α and β in $\mathbb{R}$, such that $\alpha > 0$, $\beta > 0$ and

$$\alpha \cdot \|x\|_2 \leq \|x\|_1 \leq \beta \cdot \|x\|_2 \quad \forall\, x \in X.$$

In other words, you need to prove that all the norms on finite dimensional vector spaces are equivalent.

Hint: One approach is like this.

Assume that X is n-dimensional, and assume that $e_1, \cdots, e_n$ is a basis for X (with $e_1, \cdots, e_n$ being linear independent). It is easy to show that there exists $c \in \mathbb{R}_{>0}$, such that $\|e_i\|_1 \leq c\,\|e_i\|_2$ for all $i \in \{1, \cdots, n\}$. For every given $x \in X$, we have unique $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, such that $x = \sum_{i=1}^n \lambda_i e_i$. Without loss of generality, we assume that $x \neq 0$. We can get

$$\|x\|_1 = \left\|\sum_{i=1}^n \lambda_i e_i\right\|_1 \leq \sum_{i=1}^n |\lambda_i|\,\|e_i\|_1 \leq c\sum_{i=1}^n |\lambda_i|\,\|e_i\|_2 \ .$$

Let $E = \max\{\|e_1\|_2, \cdots, \|e_n\|_2\}$. Then we have

$$\|x\|_1 \leq cE\sum_{i=1}^n |\lambda_i|.$$

Without loss of generality (else, just substitute x by $\frac{x}{\sum_{i=1}^n |\lambda_i|}$), we can assume that $\sum_{i=1}^n |\lambda_i| = 1$. Thus we get

$$\|x\|_1 \leq cE, \forall\, x \in X \ .$$

If we can show that there exists $\delta > 0$, such that

$$\|x\|_2 \geq \delta \ \text{ for all } \ x = \sum_{i=1}^{n} \lambda_i e_i \text{ with } \sum_{i=1}^{n} |\lambda_i| = 1,$$

by letting $\beta = cE/\delta$, we have

$$\|x\|_1 \leq \beta \|x\|_2 \ \text{ for all } \ x.$$

To show the existence of such δ, assume the opposite case. That is, assume $\inf_{x \in D} \|x\|_2 = 0$, where $D = \{\sum_{i=1}^{n} \lambda_i e_i \text{ with } \sum_{i=1}^{n} |\lambda_i| = 1\}$.

We can suppose $\lim_{k \to \infty} \|x_k\|_2 = 0$, where $\{x_k\}$ is a sequence in D, with

$$x_k = \sum_{i=1}^{n} \lambda_i^{(k)} e_i$$

For those $\lambda_1^{(k)}$s, note that $|\lambda_1^{(k)}| \leq 1$. Due to the compactness of $[-1, 1]$, there exists a subsequence $\{x_{k_r}\}$ of $\{x_k\}$, such that under this subsequence, $\lambda_1^{k_r}$ is convergent. By abusign the notation, we can assume that λ_1^{k} is convergent, say, it converges to λ_1. Do similiar things to coefficents of $e_2, \cdots$, until e_n, and we can eventually get an contradiction (it is your job to finish this part).

Suppose this is done, we proved that $\|x\|_1 \leq \beta \|x\|_2$ for all x. Similarly, we can prove the other direction (concerning α).